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tel: +44 1970 62 2400
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On the concept of “far points” in Hertz contact problems

Ivan Argatov¹, Mark Kachanov² and Gennady Mishuris^{3*}

¹Institut für Mechanik, Technische Universität Berlin, 10623 Berlin, Germany
ivan.argatov@gmail.com

²Department of Mechanical Engineering, Tufts University, Medford, MA 02155, USA
mark.kachanov@tufts.edu

³Department of Mathematics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK
ggm@aber.ac.uk

Abstract. Relative displacement of “far points” is used in the Hertzian contact mechanics as a measure of contact compliance. However, to be legitimate, it should be almost insensitive to the exact choice of the “far points”, and this is not always the case. The present work aims at examination of legitimacy of this concept, on specific examples of one-dimensional problem of a long rod, 2-D problem of heavy disk and 3-D problem of a sphere resting on a smooth rigid foundation. It is found that, whereas in the 1-D problem this concept may well become inadequate, in the considered 2-D and 3-D problems, the parameter controlling the legitimacy of this concept are identified and, in the vast majority of cases of practical interest, the concept is indeed legitimate. Note that the mentioned 2-D and 3-D problems are quite challenging and the presented solutions may be of interest of their own.

Keywords: Hertzian contact, far points, contact compliance, asymptotic model.

* Corresponding author

1. Introduction

A small contact area between two elastic bodies creates a highly compliant zone in the vicinity of the contact: when the bodies are pressed against one another, most of the overall deformation comes from this zone. The contact zone compliance is usually characterized by approach of two “remote points” – points on two sides of the contact that are sufficiently far from the zone (Hertz, 1881; Johnson, 1985). Relative displacement (approach) of “far points” on two sides of a contact is one of the quantities of interest in Hertzian contact problems. It is used, for example, as a measure of the contact compliance, the underlying idea being that the dominant contribution to this displacement comes from the contact zone, and the contribution of remaining parts of the contacting bodies is negligible.

For the approach of “far points” to be a legitimate measure of contact compliance, it should be almost insensitive to the exact choice of these points. This is not immediately obvious, and an example can be easily given where the concept fails. If one, or both, of the contacting bodies have elongated shapes in the direction normal to the contact plane, displacements accumulated in them may be comparable to the contribution of the contact zone. The following simple problem illustrates this statement. Consider a long elastic rod with rounded end, of length $2L$ and cross-section radius r , pressed against a rigid wall by force P applied at the opposite end; the contact with the wall is circular, of radius a (Fig. 1).

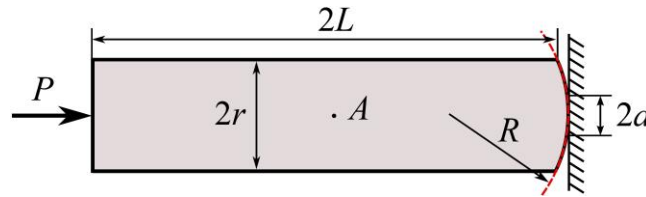


Figure 1. A long elastic rod forming contact of radius a with a rigid wall.

According to the Hertzian theory, a contribution to the approach of the rod’s center (point A) towards the wall generated by the convex contact zone equals to $\delta = \frac{1-\nu}{4a\mu}P$, where μ and ν are the shear modulus and Poisson’s ratio, respectively. We now compare δ with the displacement u_L accumulated in the rod at the distance L from the contact due

to the longitudinal deformation of the rod. For an approximate estimate, we assume, using the Saint-Venant's principle, that, somewhat away from the contact, the rod experiences uniform compression; then $u_L = \frac{L}{\pi r^2 E} P$, where E is Young's modulus. For the inequality $u_L \ll \delta$ to hold, the length L should not be too large, namely, we must have

$$\frac{a}{r} \ll c(\nu) \frac{r}{L} \quad (1.1)$$

where $c(\nu) = \pi(1-\nu^2)/2$ is a constant, which changes between 1.75 and 1.18 for ν varying from 0 to 0.5.

For example, if the aspect ratio of the rod L/r is 10, then $a \ll r/10$, i.e., practically speaking, r should be of the order of $10a$ or grater. Otherwise, the displacement from the deformation of the rod accumulated at distance L is non-negligible compared to the displacement generated by the contact zone, so that the displacement u_L at the point A cannot be used as a measure of contact compliance in the Hertzian contact theory. Observe also that $a = \sqrt{\delta R}$, and therefore, with increasing radius R of the end-rounding, the contribution from the longitudinal deformation increases as well. Note that if the “pencil-like” body is positioned vertically and is pressed against a rigid floor by gravitational forces, we have the same inequality (1.1), with somewhat different constant, of $c(\nu) = \pi(1-\nu^2)/3$.

In the above example, the failure of the concept of “far points” is related to the elongated geometry of the elastic body. However, the mentioned insensitivity to the choice of “remote points” may, possibly, be violated even for solids of non-elongated shapes. In the text to follow, we examine this issue on two example problems: a heavy 2-D disk and 3-D sphere resting on a rigid frictionless foundation that deforming under their own weight. We show that, although in most cases of practical interest the insensitivity does hold, for certain combinations of the elastic modulus and the specific weight it may be violated; these combinations will be identified in the solution obtained in the text to follow.

In a general setting, we consider two contacting bodies, and choose two points, A and A' belonging to the first one (Fig. 2) that are sufficiently far from the contact plane and thus can be regarded as “far points”. Here “sufficiently far” means that the distance from the point to the contact plane is much greater than the characteristic size of the contact area. Point A' is substantially farther away from the contact plane than A (the distance between them, in the direction normal to the plane, is comparable to the size of the contacting body). The concept of “far points” can be considered legitimate if displacements of these points in the direction towards the contact plane obey the inequality

$$\frac{|u(A') - u(A)|}{|u(A)|} \ll 1 \quad (1.2)$$

Similar inequality must hold for points B and B' of the second contacting body.

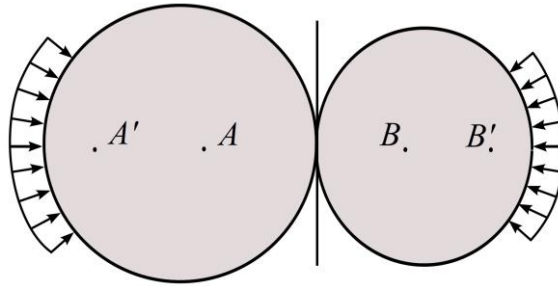


Figure 2. Two elastic bodies in contact

We examine the criterion (1.2) on two examples, a 2-D heavy elastic disk and a 3-D heavy elastic sphere that rest on a rigid frictionless foundation. Note that these problems do not seem to have been solved in literature, and may be of interest of their own. We construct, by employing the method of matched asymptotic expansions, the approximations to the displacement fields away from the contact zone. We identify the parameter that controls legitimacy of the far-points concept, which implicitly assumes that the choice of the “remote points” is unimportant, as long as they are sufficiently far from the contact, i.e. that their approach is relatively insensitive to the choice.

In particular, it will be shown that, if the points A and A' are chosen as the center of the considered body and the point at the top of it, then the following inequalities hold:

$$\frac{|u(A') - u(A)|}{|u(A)|} \leq \frac{0.76}{1.26 + \ln M} \quad (2\text{-D case}) \quad (1.3)$$

and

$$\frac{|u(A') - u(A)|}{|u(A)|} \leq \frac{0.33}{M^{1/3} - 0.73} \quad (3\text{-D case}) \quad (1.4)$$

where the numerical factors are chosen in such a way as to cover the entire interval $(0, 0.5)$ of variation of Poisson's ratio. Here, $M = \mu/(\rho_0 g R)$ is a dimensionless parameter, R is a characteristic size (radius of the disk or sphere), ρ_0 is the material density, g is the acceleration of gravity.

We use the method of matched asymptotic expansions (see, for example, van Dyke, 1964; Il'in, 1989; van Dyke, 1994) and construct the leading asymptotic terms of the solution. Note that this method has been applied earlier by Schwartz and Harper (1971) to the mathematically somewhat similar problem of compression of a circular disk by a pair of rigid circular pinches (where, however, the issue of far-points – that is of interest here – has not been examined).

In the two-dimensional problem of an elastic body compressed by two rigid bodies (punches) solved by Schwartz and Harper (1971) (this problem is also discussed in the book by Johnson (1985)), the approach of the punches serves as a straightforward measure of the local contact deformations, which from a general perspective was considered by Argatov (2001). From the point of view of mechanical work, the pair contact force/contact approach should be regarded the pair generalized force/generalized displacement. When the punches compress an elastic body, the contact force (being applied directly to one of the punches, while another is fixed) performs mechanical work on the displacement of the moving punch. In the Hertzian problem of local contact between two elastic bodies, the contact force is realized as the total of the contact pressures, which are caused by somehow applied external loading. Therefore, the question of the corresponding generalized displacement (contact approach) leads to the nontrivial problem of correct choice of the “far points”, which is simplified if this choice is insensitive to their positions (that is the case when the local contact deformation dominates the global deformation of the two contacting elastic bodies).

In Sections 2 and 3, we study respectively the 2-D (heavy disk) and the 3-D (heavy sphere) problems. Note that the conditions of validity of the “far-points” concept are substantially different in 2-D and 3-D formulations, and this justifies analysis of both problems. Mathematically, the 2-D and 3-D problems require different asymptotic constructions as well. In the text to follow, we focus on basics of the mathematical approach referring a reader to Appendices for details.

2. Heavy two-dimensional disk resting on a rigid foundation

We consider the plane strain problem of heavy elastic disk Ω of radius R that rests on a flat rigid foundation (Fig. 3) and assume that the contact is frictionless. We aim at comparing displacements of two points far from the contact zone, one of them being the disk center and another – the point at the top of the disk.

2.1. Formulation of the contact problem

We denote by Γ_c the contact interval $y_1 \in (-l, l)$ that is not known a priori. The gap between the contacting surfaces $\Delta(y_1)$ in the undeformed state is given by the equation

$y_2 = -R + \sqrt{R^2 - y_1^2}$ that can be locally approximated by the parabola

$$y_2 = -\Delta(y_1) + O(y_1^4) \quad \text{where} \quad \Delta(y_1) = \frac{y_1^2}{2R} \quad (2.1)$$

with the Hooke’s law

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \delta_{ij}\mu\frac{2\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}) \quad (2.2)$$

and equilibrium equations

$$\frac{\partial \sigma_{i\alpha}}{\partial x_\alpha} + \delta_{i2}\gamma_0 = 0 \quad (2.3)$$

(repeated Greek symbols indicate summation from 1 to 2) to be satisfied where $\gamma_0 = \rho_0 g$ is the specific 2-D weight of the disk material.

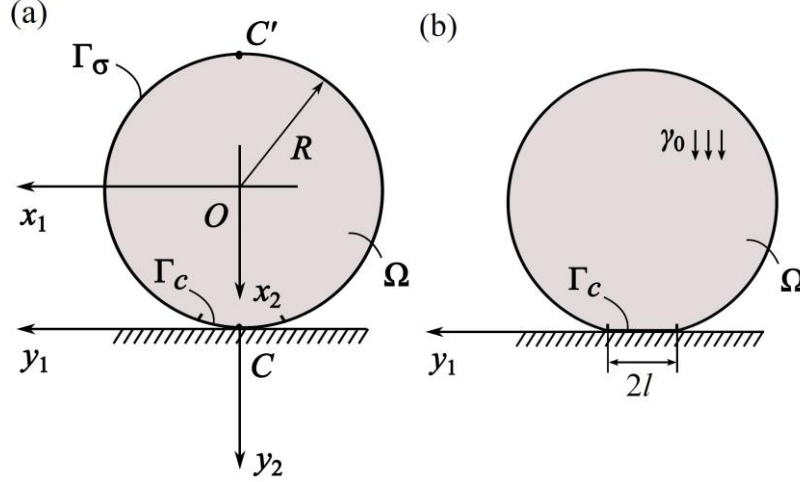


Figure 3. Heavy circular elastic disk on a rigid frictionless foundation:
a) Undeformed configuration; b) Loaded configuration

The following conditions must be satisfied in the (yet unknown) contact interval $y_1 \in (-l, l)$:

$$u_n \leq \Delta \text{ (non-penetration)} \quad (2.4)$$

$$\sigma_n \leq 0 \quad \text{(the normal traction is compressive)} \quad (2.5)$$

$$(u_n - \Delta) \sigma_n = 0 \quad \text{(at least one of the multipliers must be zero)} \quad (2.6)$$

$$\sigma_t = 0 \quad \text{(no friction)} \quad (2.7)$$

We introduce parameter γ_0^* by the relation

$$\gamma_0 = \varepsilon \gamma_0^* \quad (2.8)$$

where $\varepsilon > 0$ is a small parameter.

2.2. Far-field asymptotics

In order to construct the far-field asymptotic approximation $\mathbf{v}(\mathbf{x})$ for the displacement field $\mathbf{u}(\mathbf{x})$, we consider an auxiliary problem where the distributed normal traction in the contact zone is replaced by a unit point force and the center of the disk is fixed. Though the influence of the contact zone is modelled by the action of a concentrated reaction force, the far-field in the actual problem is approximated in the leading-order by the far-field in the auxiliary problem. Its solution (singular at the point of application of the force) is denoted by $\mathbf{G}(\mathbf{x})$; it is subject to the condition

$$G(O) = 0 \quad (2.9)$$

and is given by the following formulas (see, for example, Barber, 2002):

$$\begin{aligned} 4\pi\mu G_1(\mathbf{x}) &= -2(1-2\nu)\psi_1 + \sin 2\psi_1 + (1-2\nu)\frac{r}{R}\sin\varphi - \frac{r^2}{2R^2}\sin 2\varphi \\ -4\pi\mu G_2(\mathbf{x}) &= 4(1-2\nu)\ln\frac{R}{r_1} + 2\cos^2\psi_1 + (1-2\nu)\frac{r}{R}\cos\varphi + \frac{r^2}{2R^2}(1+2\sin^2\varphi) - 2 \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2}, \quad r_1 = \sqrt{x_1^2 + (R + x_2)^2}, \quad \sin\varphi = \frac{x_1}{r}, \quad \cos\varphi = \frac{x_2}{r}, \\ \psi_1 &= \arcsin \frac{x_1}{\sqrt{x_1^2 + (R + x_2)^2}} \end{aligned}$$

Taking asymptotics of (2.10) near point C yields the following expansion (see, for example, Argatov, 2001):

$$\mathbf{G}(\mathbf{x}) = \mathbf{S}(y/R) + \mathcal{G}A_0\mathbf{e}_2 + O(|y|), \quad |y| \rightarrow 0 \quad (2.11)$$

Here, $\mathbf{S}(\zeta)$ is the solution of Flamant's problem for the half-plane $\zeta_2 \leq 0$ loaded by unit point force in the $-\zeta_2$ direction, i.e.

$$4\pi\mu S_1(\zeta) = -\frac{2\zeta_1\zeta_2}{|\zeta|^2} + 2(1-\nu)\arctan\frac{\zeta_1}{\zeta_2}, \quad 4\pi\mu S_2(\zeta) = 4(1-\nu)\ln|\zeta| - \frac{2\zeta_2^2}{|\zeta|^2} \quad (2.12)$$

with $\zeta = (\zeta_1, \zeta_2)$ being dimensionless coordinates along the x_1, x_2 axes normalized to the radius R , and

$$A_0 = \frac{5-4\nu}{8(1-\nu)}, \quad \mathcal{G} = \frac{1-\nu}{4\pi\mu} \quad (2.13)$$

Note that, for an arbitrary positive constant A , function $\mathbf{S}(\zeta)$ has the following property:

$$\mathbf{S}(A\zeta) = \mathbf{S}(\zeta) + \frac{(1-\nu)\ln A}{\pi\mu}\mathbf{e}_2 \quad (2.14)$$

Having in view the original contact problem, we focus on the far-field asymptotics of the displacement field in the form of solution to the auxiliary problem of an elastic disk equilibrated by a point force (Fig. 4), that is

$$\mathbf{v}(\mathbf{x}) = P\mathbf{G}(\mathbf{x}) + \alpha\mathbf{e}_2 \quad (2.15)$$

Here, α is the (unknown yet) vertical displacement of the disk center (note that its value cannot be found from the 2D Hertzian theory), P is the contact force that is found from the following equilibrium condition:

$$P = \gamma_0 s \quad (2.16)$$

($s = \pi R^2$ is the disk area). With the account of (2.8), we rewrite the above formula as

$$P = \varepsilon P^* \quad (2.17)$$

where $P^* = \gamma_0^* s$. It is later shown that the force scaling (2.17) implies the following scaling for the relative approach:

$$\alpha = \varepsilon \alpha^* \quad (2.18)$$

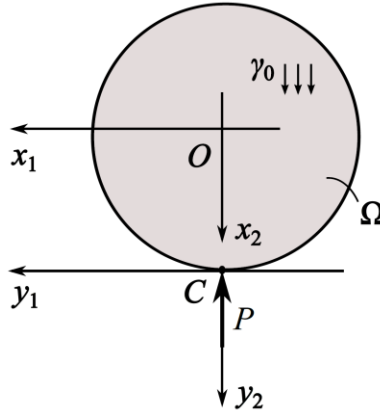


Figure 4. Heavy elastic disk equilibrated by a concentrated force

2.3. The boundary layer approximation

We introduce “stretched” coordinates $\boldsymbol{\eta} = (\eta_1, \eta_2)$ as follows:

$$\eta_i = \varepsilon^{-1/2} y_i \quad (2.19)$$

where the stretching factor $\varepsilon^{-1/2}$ is chosen in such a way that the size of the contact region predicted by the Hertzian theory (see formulas (2.31) and (2.33) to follow), as expressed in coordinates $\boldsymbol{\eta} = (\eta_1, \eta_2)$, does not depend on parameter ε . In these coordinates, the gap (2.1) has the form

$$\Delta = \varepsilon \frac{\eta_1^2}{2R} + O(\varepsilon^2 \eta_1^4) \quad (2.20)$$

and endpoints of the arc Γ_c are shifted on distance $\varepsilon^{-1/2}l$ from point C . This allows one to reformulate the problem of the near-field asymptotic approximation $\mathbf{w}(\boldsymbol{\eta})$ of the displacement $\mathbf{u}(\mathbf{x})$ as a boundary-layer problem for a semi-infinite domain with the parabolic boundary defined by the following equation containing small parameter $\varepsilon^{1/2}$ (see (2.1) and (2.20)):

$$\eta_2 = -\varepsilon^{1/2} \frac{\eta_1^2}{2R}, \quad \eta_1 \in (-\infty, \infty) \quad (2.21)$$

According to the method of matched asymptotic expansions, formulas (2.11) and (2.15) determine behavior of the vector function $\mathbf{w}(\boldsymbol{\eta})$ at infinity. Thus, letting $|\boldsymbol{\eta}| \rightarrow \infty$ and ignoring terms $O(\varepsilon^{1/2}/|\boldsymbol{\eta}|)$ in (2.11), we obtain

$$\mathbf{w}(\boldsymbol{\eta}) = \varepsilon P^*[\mathbf{S}(\varepsilon^{1/2}\boldsymbol{\eta}/R) + \mathcal{G}A_0\mathbf{e}_2] + \varepsilon \alpha^* \mathbf{e}_2 + O(|\boldsymbol{\eta}|^{-1}) \quad (2.22)$$

where the normalization relations (2.17) and (2.18) are used.

Utilizing conditions (2.22) and (2.14), the vector-function $\mathbf{w}(\boldsymbol{\eta})$ that satisfies the homogeneous equilibrium equations (no body force), can be written in the form

$$\mathbf{w}(\boldsymbol{\eta}) = \varepsilon \mathbf{W}(\boldsymbol{\eta}) + \varepsilon \left[P^* \left(\mathcal{G}A_0 + \ln \sqrt{\varepsilon} \right) + \alpha^* \right] \mathbf{e}_2 \quad (2.23)$$

that involves new vector-function $\mathbf{W}(\boldsymbol{\eta})$ satisfying the asymptotic condition at infinity

$$\mathbf{W}(\boldsymbol{\eta}) = P^* \mathbf{S}(\boldsymbol{\eta}/R) + O(|\boldsymbol{\eta}|^{-1}), \quad |\boldsymbol{\eta}| \rightarrow \infty \quad (2.24)$$

and, in view of (2.12), possesses the logarithmic behavior at infinity.

Finally, we derive boundary conditions for $\mathbf{W}(\boldsymbol{\eta})$ in the stretched coordinate system. Substituting expressions (2.19), (2.20) and (2.23) into the boundary conditions of unilateral frictionless contact (2.4)–(2.7), we obtain

$$W_2(\eta_1, 0) \leq \Delta_\varepsilon^*(\eta_1), \quad \sigma_{22}(\mathbf{W}, \eta_1, 0) \leq 0, \quad \left[W_2(\eta_1, 0) - \Delta_\varepsilon^*(\eta_1) \right] \sigma_{22}(\mathbf{W}, \eta_1, 0) = 0 \quad (2.25)$$

$$\sigma_{21}(\mathbf{W}, \eta_1, 0) = 0, \quad \eta_1 \in (-\infty, \infty) \quad (2.26)$$

where it is denoted

$$\Delta_\varepsilon^*(\eta_1) = \frac{\eta_1^2}{2R} - P^* \left(\mathcal{G}A_0 + \ln \sqrt{\varepsilon} \right) - \alpha^* \quad (2.27)$$

Relations (2.24) – (2.26) plus the homogeneous elasticity equations in half-plane $\eta_2 < 0$ constitute the boundary-layer problem of unilateral contact for the vector-function $\mathbf{W}(\boldsymbol{\eta})$.

2.4. Near-field asymptotics

We represent vector-function $\mathbf{W}(\boldsymbol{\eta})$ in the form (see, for example, Johnson, 1985 or Muskhelishvili, 1963):

$$\mathbf{W}(\boldsymbol{\eta}) = \int_{-l_*}^{l_*} \mathbf{S}\left(\frac{\eta_1 - \xi}{R}, \frac{\eta_2}{R}\right) p^*(\xi) d\xi \quad (2.28)$$

where $l_* = l/\sqrt{\varepsilon}$ is the half-length of the (unknown yet) contact interval in the stretched coordinates (Fig. 5) and $p^*(\eta_1)$ is pressure in the contact region given by

$$p^*(\eta_1) = \frac{2P^*}{\pi l_*} \sqrt{1 - \frac{\eta_1^2}{l_*^2}} \quad (2.29)$$

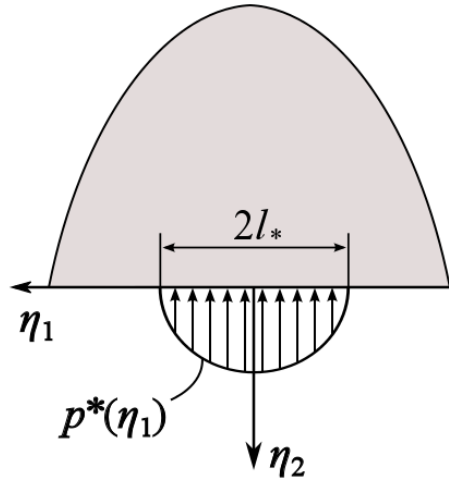


Figure 5. Elastic half-plane loaded with the Hertzian contact pressure

The vector-function (2.28) satisfies the asymptotic condition (2.24), the boundary condition (2.26) of frictionless contact, and the following ones:

$$\begin{aligned} \sigma_{22}(\mathbf{W}, \eta_1, 0) &= 0, \quad |\eta_1| \geq l_* \\ W_2(\eta_1, 0) &= \mathcal{G}P^* \left(\frac{\eta_1^2}{l_*^2} - \ln \frac{2R}{l_*} - \frac{1}{2} \right), \quad |\eta_1| < l_* \end{aligned} \quad (2.30)$$

Substituting the boundary values (2.30) and the expression for $\Delta_\varepsilon^*(\eta_1)$ from (2.27) into the displacement compatibility equation $W_2(\eta_1, 0) = \Delta_\varepsilon^*(\eta_1)$ at $|\eta_1| < l_*$, we arrive, after some algebra, at the following system of non-linear algebraic equations for l_* and α^* :

$$l_*^2 = 2RP^*\mathcal{G} \quad (2.31)$$

$$\mathcal{G}P^*\left(\ln \frac{2R}{l_*\sqrt{\varepsilon}} + \frac{1}{2} - A_0\right) = \alpha^* \quad (2.32)$$

Returning from the stretched scale to the original one, in accordance with (2.19), we obtain the actual half-length of the contact interval: $l = l_*\sqrt{\varepsilon}$. The meaning of parameter α^* is clarified in the text to follow.

2.5. Analysis of the “far-points” displacements

Returning to the original coordinates and eliminating the auxiliary parameter ε we obtain, with the account of results given above, the contact pressure

$$p(y_1) = \frac{2P}{\pi l} \sqrt{1 - \frac{y_1^2}{l^2}}, \quad \text{where} \quad l^2 = 2\mathcal{G}RP \quad (2.33)$$

The constant α entering the far-field expression (2.12) is given by

$$\alpha = \mathcal{G}P\left(\ln \frac{2R}{l} + \frac{1}{2} - A_0\right) \quad (2.34)$$

with the contact force P determined from Eq. (2.16).

Having considered the far-field asymptotics, we now focus on the main issue – the approach of the disk towards the support plane. We examine the sensitivity of the said approach to a particular choice of the far point, by comparing results for two different far points. We choose the two points to be quite far apart: one of them is the disk center and another one – the point at the top of the disk.

For the center of the disk, we obtain, with the account of the normalization condition (2.9) and relations (2.34),

$$u_2(O) = PG_2(O) + \alpha = \mathcal{G}P\left(\ln \frac{2R}{l} + \frac{1}{2} - A_0\right) \quad (2.35)$$

For the point at the top of the disk, we obtain, making use of formulas (2.10) and (2.34),

$$u_2(C') = PG_2(C') + \alpha = \mathcal{G}P \left(\ln \frac{2R}{l} + \frac{1}{2} - A_0 + K' \right) = u_2(O) + (\mathcal{G}P)K' \quad (2.36)$$

where the following constant is introduced:

$$K' = \ln 2 - \frac{3 - 2\nu}{8(1 - \nu)} \quad (2.37)$$

Thus, we arrive at the following key result:

$$\frac{u_2(C') - u_2(O)}{u_2(O)} = \frac{2K'}{\ln M + \ln(8/(1 - \nu)) - 1/(4 - 4\nu)} \quad (2.38)$$

The above quantity is controlled by the following dimensionless parameter:

$$M = \frac{\mu}{\gamma_0 R} \quad (2.39)$$

For the ratio (2.38) to be small – and hence for the notion of “far points” to be legitimate – the value of M must be sufficiently large. For example, at $\nu = 0.3$ (the sensitivity to ν is relatively low), for the ratio to be smaller than $1/10$ we must have $\mu/(\gamma_0 R) > 12$. For μ and γ_0 typical for metals, this means that the disk radius should not exceed a very large value, of the order of kilometers – this limitation is irrelevant for typical engineering problems. However, for materials that are very soft elastically, this value may be orders of magnitude smaller, and this limitation may be relevant.

3. Three-dimensional problem of a heavy sphere resting on a rigid foundation

We now examine the validity of the far-points concept in the 3-D setting, aiming at obtaining a 3-D counterpart of the relation (2.38).

3.1. Formulation of the problem

In the spherical coordinate system ρ, θ, φ centered at the sphere center, the support plane is defined by the equation

$$\rho = \frac{R}{\cos \theta} \quad (3.1)$$

(R is the radius of the sphere). The point of contact C in the undeformed state is taken as the origin of the local coordinate system $y_1 = -x_2$, $y_2 = -x_1$, $y_3 = R - x_3$ (Fig. 6). The density of the body force is $\gamma_0 \mathbf{e}_3$, where $\gamma_0 = \rho_0 g$ and ρ_0 is the material density.

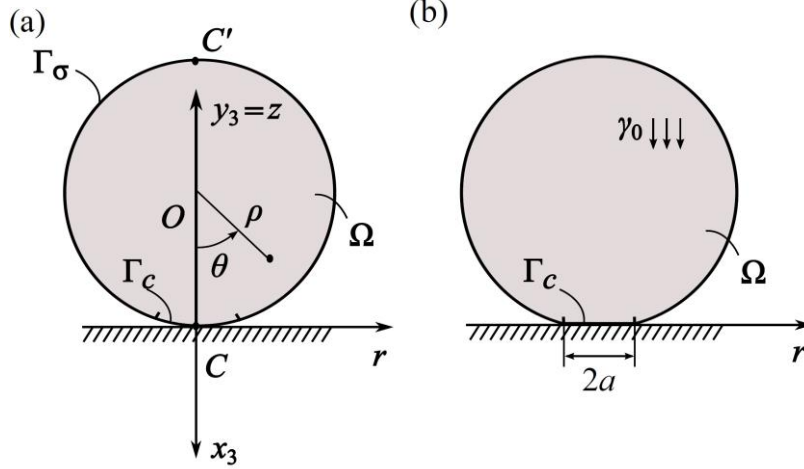


Figure 6. Heavy elastic sphere on a rigid frictionless support:

a) Undeformed configuration; b) Loaded configuration

We denote by Γ_c a part (not known a priori) of the boundary $\partial\Omega$ that comes in contact during the deformation. Near point C , the surface Γ_c can be approximated by the paraboloid

$$y_3 = \frac{y_1^2 + y_2^2}{2R} \quad (3.2)$$

The displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ must satisfy Lamé equations in Ω , the traction free boundary condition in the remainder of $\partial\Omega$, and boundary conditions (2.4)–(2.7) on Γ_c where \mathbf{n} is the unit outward normal to the sphere $\partial\Omega$, σ_n is the normal stress, and σ_t is the shear stress vector. In accordance with formulas (3.1) and (3.2), the gap is

$$\Delta(y_1, y_2) = \frac{R^2}{\sqrt{R^2 - y_1^2 - y_2^2}} - R \cong \frac{y_1^2 + y_2^2}{2R} \quad (3.3)$$

Similarly to the 2-D problem, we introduce a small positive parameter ε , set $\gamma_0 = \varepsilon \gamma_0^*$ (see formula (2.8) and make use of the method of matched asymptotic expansions.

3.2. Far-field asymptotics

We introduce the auxiliary problem (similar to the one considered in the 2-D setting): a unit point force $-\mathbf{e}_3$ applied at contact point C is equilibrated by a distributed gravity force $V^{-1}\mathbf{e}_3$ (where $V=(4\pi/3)R^3$ is the sphere volume), and denote by $\mathbf{G}(\mathbf{x})$ the displacement field in this problem assuming that

$$\mathbf{G}(\mathbf{O})=0 \quad (3.4)$$

Using the results of Sternberg and Rosenthal (1952), the following asymptotic expansion was obtained by Argatov (2006):

$$\mathbf{G}(\mathbf{x}) = \mathbf{T}(\mathbf{y}) + \mathbf{t}(\mathbf{y}) + 2\mathcal{G}A_0\mathbf{e}_3 + O(|\mathbf{y}| \ln(|\mathbf{y}|/R)), \quad |\mathbf{y}| \rightarrow 0 \quad (3.5)$$

Here, $\mathbf{T}(\mathbf{y})$ is the solution of Boussinesq's problem of loading of an elastic half-space by a unit point force, $|\mathbf{y}| = \sqrt{y_1^2 + y_2^2 + y_3^2}$ and the elastic constant \mathcal{G} is defined by (2.13). As far as the constant A_0 (that, in the 3-D case, has the dimension of length) is concerned, its value cannot be readily determined using the results of Sternberg and Rosenthal (1952) for an elastic sphere under point forces. Instead, its value can be found using results of Bondareva (1969) for a heavy elastic sphere equilibrated by a point force. However, for our purposes, it is simpler to use the asymptotic solution of the contact problem constructed by Argatov (2005) using Bondareva's integral equation representation (see Appendix A).

In the cylindrical coordinate system $r = \sqrt{y_1^2 + y_2^2}$, $z = y_3$, Boussinesq's solution takes the form

$$4\pi\mu T_r(r, z) = \frac{rz}{(r^2 + z^2)^{3/2}} - (1-2\nu) \frac{\sqrt{r^2 + z^2} - z}{r\sqrt{r^2 + z^2}} \quad (3.6)$$

$$4\pi\mu T_z(r, z) = \frac{2(1-\nu)}{\sqrt{r^2 + z^2}} + \frac{z^2}{(r^2 + z^2)^{3/2}} \quad (3.7)$$

The second term in the asymptotic expansion (3.5) accounts for curvature of the sphere and has the logarithmic behavior as $|\mathbf{y}| \rightarrow 0$:

$$\frac{4\pi\mu R}{1-2\nu} t_r(r, z) = -2(1-2\nu) \frac{r}{\sqrt{r^2 + z^2} + z} + \frac{rz}{\sqrt{r^2 + z^2} (\sqrt{r^2 + z^2} + z)}$$

$$\frac{4\pi\mu R}{1-2\nu} t_z(r, z) = \frac{z}{\sqrt{r^2 + z^2}} - (1-2\nu) \ln \frac{\sqrt{r^2 + z^2} + z}{2R} \quad (3.8)$$

We now return to the original problem, of a heavy sphere on a rigid half-space. The far-field asymptotics of the displacement field has the form

$$\mathbf{v}(\mathbf{x}) = P\mathbf{G}(\mathbf{x}) + \alpha\mathbf{e}_3 \quad (3.9)$$

where α is the vertical displacement of the sphere center. The contact force P must satisfy the equilibrium condition

$$P = \gamma_0 V \quad (3.10)$$

In accordance with the normalization condition (2.8), we have

$$P = \varepsilon P^* \quad (3.11)$$

The Hertzian theory predicts that $P \sim \alpha^{3/2}$. Thus, in view of (3.11), we set

$$\alpha = \varepsilon^{2/3} \alpha^*. \quad (3.12)$$

3.3. Boundary layer formulation near the contact

Motivated by relations (3.11) and (3.12) we use the stretched coordinates, with the stretching factor $\varepsilon^{-1/3}$ as follows:

$$\mathbf{y} = \varepsilon^{1/3} \boldsymbol{\eta}, \quad \boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \quad (3.13)$$

With the account of relations (3.10) and (3.14), we have

$$\mathbf{v}(\mathbf{x}) = P(\mathbf{T}(\mathbf{y}) + \mathbf{t}(\mathbf{y}) + 2\mathcal{A}_0\mathbf{e}_3) + \alpha\mathbf{e}_3 + O(|\mathbf{y}| \ln(|\mathbf{y}|/R)) \quad (3.14)$$

where vectors $\mathbf{T}(\mathbf{y})$ and $\mathbf{t}(\mathbf{y})$ are defined by Eqs. (3.6) – (3.8). Formulating (3.14) in the stretched coordinates and taking into account (3.11) and (3.12), we obtain:

$$\begin{aligned} \mathbf{v}(\mathbf{x})|_{\mathbf{y}=\varepsilon^{1/3}\boldsymbol{\eta}} &= \varepsilon^{2/3} \left(P^* \mathbf{T}(\boldsymbol{\eta}) - \alpha^* \mathbf{i}_3 \right) \\ &+ \varepsilon P^* (\mathbf{t}(\boldsymbol{\eta}) - 2\mathcal{G}(C_\varepsilon + A_0) \mathbf{i}_3) + O\left(\varepsilon^{4/3} |\boldsymbol{\eta}| \ln\left(\varepsilon^{1/3} |\boldsymbol{\eta}|/R\right)\right) \end{aligned} \quad (3.15)$$

where \mathbf{i}_3 is the unit vector of the η_3 axis and $C_\varepsilon = (1-2\nu)^2 (1-\nu)^{-1} R^{-1} \ln \varepsilon^{1/3}$.

The near-field asymptotics of the displacement field in the vicinity of the contact will be denoted by $\mathbf{w}(\boldsymbol{\eta})$. The condition of matching the vector-function $\mathbf{w}(\boldsymbol{\eta})$ with the far-field, $\mathbf{v}(\mathbf{x})$, according to the leading terms in (3.15), implies the following matching condition:

$$\mathbf{w}(\boldsymbol{\eta}) = \varepsilon^{2/3} \left(P^* \mathbf{T}(\boldsymbol{\eta}) - \alpha^* \mathbf{i}_3 \right) + O(|\boldsymbol{\eta}|^{-2}), \quad |\boldsymbol{\eta}| \rightarrow \infty \quad (3.16)$$

Hence the vector-function $\mathbf{w}(\boldsymbol{\eta})$ can be represented in the form

$$\mathbf{w}(\boldsymbol{\eta}) = \varepsilon^{2/3} \mathbf{W}(\boldsymbol{\eta}) - \varepsilon^{2/3} \alpha^* \mathbf{i}_3 \quad (3.17)$$

where, in view of (3.16), the vector $\mathbf{W}(\boldsymbol{\eta})$ satisfies the following condition at infinity:

$$\mathbf{W}(\boldsymbol{\eta}) = P^* \mathbf{T}(\boldsymbol{\eta}) + O(|\boldsymbol{\eta}|^{-2}), \quad |\boldsymbol{\eta}| \rightarrow \infty \quad (3.18)$$

Finally, the boundary conditions of unilateral contact for $\mathbf{W}(\boldsymbol{\eta})$, written in the stretched coordinates, have the form

$$\begin{aligned} W_3(\eta_1, \eta_2, 0) &\geq \Delta^*, \quad \sigma_{33}(\mathbf{W}; \eta_1, \eta_2, 0) \leq 0, \quad [W_3(\eta_1, \eta_2, 0) - \Delta^*] \sigma_{33}(\mathbf{W}; \eta_1, \eta_2, 0) = 0 \\ \sigma_{31}(\mathbf{W}; \eta_1, \eta_2, 0) &= \sigma_{32}(\mathbf{W}; \eta_1, \eta_2, 0) = 0 \end{aligned} \quad (3.19)$$

in the entire plane (η_1, η_2) . Here, in view of (3.3) and (3.12), we denoted

$$\Delta^*(\eta_1, \eta_2) = \alpha^* - \frac{\eta_1^2 + \eta_2^2}{2R}$$

The unilateral boundary conditions (3.19), the asymptotic condition (3.18), and Lamé equations constitute the boundary-layer problem for the vector-function $\mathbf{W}(\boldsymbol{\eta})$ that determines the near-field (3.17).

3.4. Near-field asymptotics. The leading order approximation

We represent the boundary-layer function $\mathbf{W}(\boldsymbol{\eta})$ in the form

$$\mathbf{W}(\boldsymbol{\eta}) = \iint_{\sqrt{\eta_1^2 + \eta_2^2} < a^*} \mathbf{T}(\eta_1 - \xi_1, \eta_2 - \xi_2, \eta_3) p^*(\xi_1, \xi_2) d\xi_1 d\xi_2$$

where, in accordance with the Hertzian theory, the contact pressure is

$$p^*(\xi_1, \xi_2) = p_0^* \sqrt{1 - \frac{\eta_1^2 + \eta_2^2}{a_*^2}}$$

Its integration over the contact area yields the contact force

$$P^* = \frac{2}{3} p_0^* \pi a_*^2 \quad (3.20)$$

The contact radius a_* and the contact approach α_* are determined as follows:

$$a_* = \left(\frac{3\pi}{2} \mathcal{G} P^* R \right)^{1/3} \quad (3.21)$$

$$\alpha_* = \frac{a_*^2}{R} = \left(\frac{9\pi^2 \mathcal{G}^2 (P^*)^2}{4R} \right)^{1/3} \quad (3.22)$$

Returning now to the original coordinates, we obtain, with the account of Eqs. (3.10), (3.11), and (3.20)–(3.22), that

$$\alpha = \left(\frac{\pi(1-\nu)}{2\mu} \gamma_0 R \right)^{2/3} \quad (3.23)$$

Formula (3.23) represents the displacement of the disk center (point O) – one of the two “distant” points in the considered problem. However, this is the Hertzian approximation only (that is the leading asymptotic solution), and, evidently, it does not depend on the shape of elastic body outside the region of local perturbations. In other words, the displacement of the point C' – another of the two “distant” points – coincides with that given by formula (3.23).

3.5. Relative approach of the sphere — an asymptotic model

Observe that formula (3.23) follows from the Hertzian formula (3.22) and the equilibrium equation (3.10). Therefore, to obtain a correction to formula (3.23), it is necessary to refine formula (3.22) by taking into account the influence from the far-field.

In Appendix A, we formulate the contact problem under consideration in the form of an integral equation, and derive the first-order correction to the Hertzian equations. In particular, the following approximate formula for the vertical displacement of the center of the sphere as a function of the applied load holds:

$$\frac{\alpha}{R} = \left(\frac{3\pi \mathcal{G}}{2R^2} P \right)^{2/3} + \frac{P}{8\pi R^2 \mu} \left[\frac{2}{3} \beta^2 \ln \frac{R^2}{12\pi \mathcal{G} P} + C'_0 \right] \quad (3.24)$$

where

$$C'_0 = 2c_0 + \frac{19}{6} (1-2\nu)^2 + \frac{3}{2} \frac{1-2\nu}{1-\nu}$$

Alternatively, we can express P in terms of α as follows:

$$P = \frac{2\sqrt{R}}{3\pi g} \alpha^{2/3} \left\{ 1 + \frac{1}{6\pi^2 g_1} \sqrt{\frac{\alpha}{R}} \left[2\beta^2 \ln \left(\frac{1}{2} \sqrt{\frac{R}{\alpha}} \right) + C'_0 \right] \right\}^{-1} \quad (3.25)$$

For small values of the ratio $\gamma_0 R/\mu$, the Hertzian result (3.22) is recovered from Eq.(3.25) by neglecting the second term in the braces.

We now focus on the quantity of central interest — the difference between vertical displacements of points O (the center of the sphere) and C' (the top point). Formulas (3.4), (3.9), and (3.24) yield

$$\begin{aligned} u_3(O) &= PG_3(O) + \alpha \\ &= \left(\frac{3\pi g}{2\sqrt{R}} P \right)^{2/3} + \frac{P}{8\pi R \mu} \left[\frac{2}{3} \beta^2 \ln \frac{R^2}{12\pi g P} + C'_0 \right] \end{aligned} \quad (3.26)$$

Taking into account (A.2), (A.3) and (3.24), we obtain

$$\begin{aligned} u_3(C') &= PG_3(C') + \alpha \\ &= \left(\frac{3\pi g}{2\sqrt{R}} P \right)^{2/3} + \frac{P}{8\pi R \mu} \left[\frac{2}{3} \beta^2 \ln \frac{R^2}{12\pi g P} + C'_0 + K' \right] \end{aligned} \quad (3.27)$$

where

$$K' = 2(1-\nu) - \frac{1-2\nu}{2(1+\nu)} - \operatorname{Re} \left(A_r \frac{2-m}{3-m} \right)$$

From Eqs. (3.10) and (3.26), (3.27), we finally arrive at

$$\frac{u_3(C') - u_3(O)}{u_3(O)} = \frac{M^{-1/3} K'}{3 \cdot 2^{1/3} (\pi(1-\nu))^{2/3} + M^{-1/3} \left[\frac{2}{3} \beta^2 \ln \left(\frac{M}{4\pi(1-\nu)} \right) + C'_0 + K' \right]} \quad (3.28)$$

Note that, similarly to the 2-D problem for a heavy elastic disk, the key role here is played by the dimensionless parameter $M = \mu/(R\gamma_0)$. For the ratio (3.28) to be small – and the concept of “far point” to be legitimate – the parameter M should be sufficiently large. As seen from comparison of (3.28) with the similar relation (2.38) in the 2-D case, this condition is substantially different mathematically. The values of constants K' and $C'_0 + K'$ for several values of Poisson’s ratio and other results are given in Table 1. The dimensional quantity μ/γ_0 is given in Table 2 for several materials.

4. Discussion and conclusions

Fig. 7 shows the relative displacements ratio as a function of $1/M$ for both cases. It is seen that, as a rule, the ratio is, indeed, quite small. We add, however, that the ratio is much larger in the 2-D problem, so that one has to be careful in using the concept of far points. This observation appears to be consistent with the fact that, in the 1-D problem of a long rod (considered in Section 1), the concept of “far points” may become inadequate quite easily.

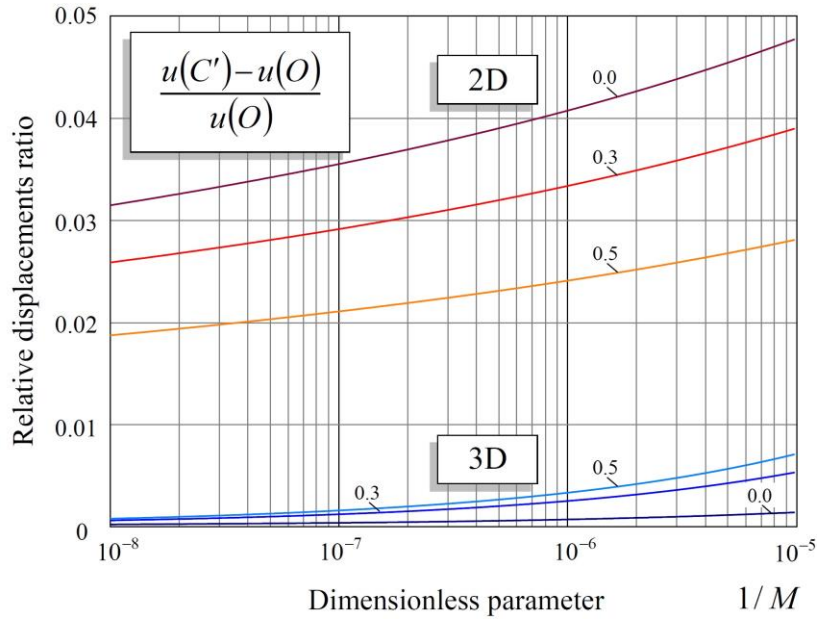


Figure 7. The dependence of the ratio for the relative displacements $[u_n(C') - u_n(O)]/u_n(O)$, $n = 2, 3$, as a function of $1/M$ in the 2-D and 3-D cases, respectively, obtained for different Poisson's ratios.

An interesting observation is that, as seen from Fig. 8, the mentioned ratio has different dependencies on Poisson's ratio in the 2-D and 3-D problems.

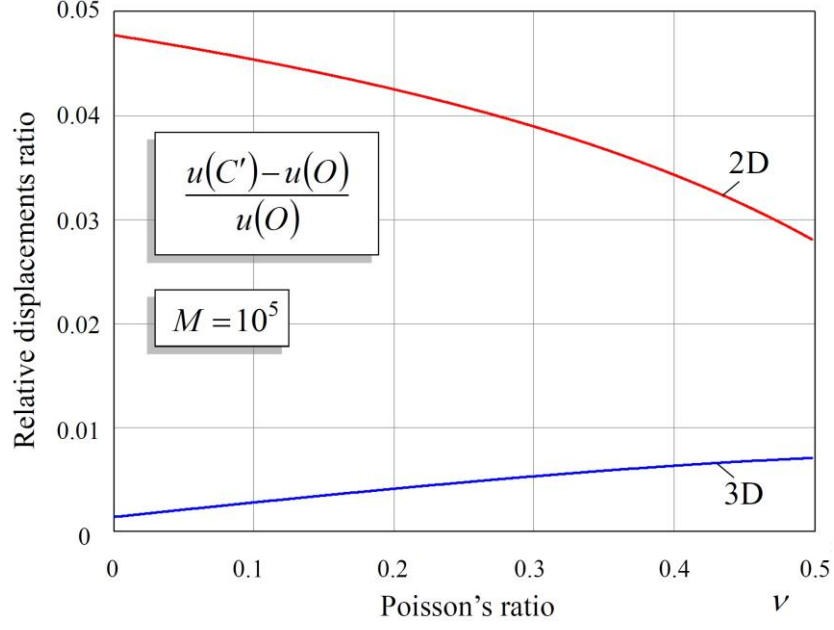


Figure 8. The dependence of the ratio $[u_n(C') - u_n(O)]/u_n(O)$, $n = 2, 3$ on Poisson's ratio

Observe that the discriminating parameter $M = \mu/(\rho_0 g R)$ is composed of two material characteristics, μ and ρ_0 , one geometrical parameter, R , which represents both the local curvature at the contact and the body size in the normal direction, and one environmental force field parameter, g , (for Earth conditions $g = 9.81 \text{ m/s}^2$). Therefore, by increasing the parameter g (by means of accelerating the support or by superimposing a static field, e.g., magnetic field), one can easily decrease the value of M . This simple consideration shows that the effect of choice of “far points” will be more essential in the Hertzian theory of quasi-static impact than in the equilibrium problem involving self-weight.

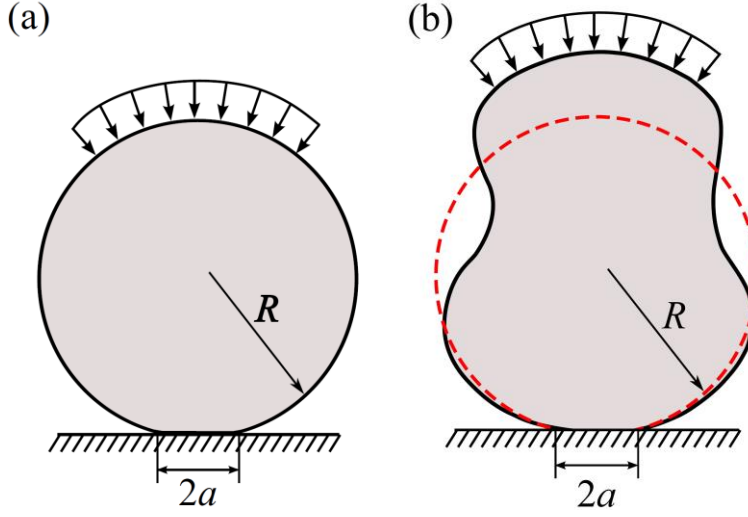


Figure 9. Elastic bodies pressed against a rigid frictionless foundation (the same total-force loading): (a) Elastic sphere of radius R ; (b) Axisymmetric elastic body with local curvature radius R .

The above-developed approach can be applied to other contact configurations as well. For instance, let us consider an elastic sphere of radius R pressed against frictionless rigid foundation by an axisymmetric system of surface tractions (Fig. 9a), generating circular contact area of radius a ; the sphere center obtains vertical displacement α . Hertzian theory, having local character, will predict the same contact parameters a and α for any other axisymmetric elastic body with the same local curvature radius R (Fig. 9b) under the condition that the two bodies have the same elastic constant \mathcal{G} and the total applied force P is the same:

$$\alpha = \frac{a^2}{R} = \left(\frac{9\pi^2 \mathcal{G}^2 P^2}{4R} \right)^{1/3}$$

In light of our results, relating the value of α to the discussed ratio will require knowledge of Green's function $\mathbf{G}(\mathbf{x})$ for the new considered configuration (Fig. 10). Note also that the applicability of the Hertz theory to non-small contact areas was considered by Zhupanska (2011) on the example of contact problem for elastic spheres subjected to the concentrated forces applied in their centers.

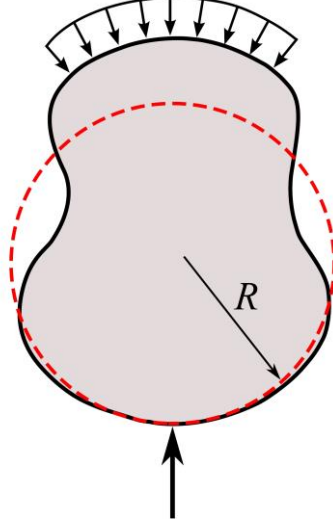


Figure 10. Axisymmetric elastic body, with an applied external loading equilibrated by the point force reaction generated by the support

The level of the stress state around the Hertzian contact spot is governed by the value of the maximum contact pressure, p_0 . In the two-dimensional case, according to the known solution (Johnson, 1985), we have

$$p_0 = \sqrt{\frac{PE^*}{\pi R}}$$

Taking into account the relations $E^* = 2\mu/(1-\nu)$ and $P = \gamma_0\pi R^2$, we readily get

$$\frac{p_0}{\mu} = \sqrt{\frac{2}{(1-\nu)M}}$$

The maximum principal shear stress is given by $(\tau_1)_{\max} = 0.30p_0$, so that the corresponding maximum shear strain will be $(\gamma_1)_{\max} = 0.42/\sqrt{(1-\nu)M}$. Thus, $(\gamma_1)_{\max}$ is inversely proportional to \sqrt{M} , and for materials that are very soft elastically, the values of local strains may overcome the level of small deformations. For instance, by assuming that $(\gamma_1)_{\max} \leq 0.05$, we get an estimate $M \geq 72/(1-\nu)$. By substituting the obtained lower bound into the right-hand side of (2.38), we get that the quantity $\frac{u_2(C') - u_2(O)}{u_2(O)} \times 100\%$ decreases from 10% for $\nu = 0$ to 5% for $\nu = 0.5$. Thus, this conservative estimate shows that the “far points” effect may exhibit itself even in the

range of relatively small deformations. The analogous considerations can be also carried out in the three-dimensional case to show that in the 3D case the “far points” effect is weaker in the special case under consideration (of a heavy sphere on a rigid support).

The presented analysis is related to elastic bodies of finite sizes, whereas the sensitivity to the choice of the “far points” dramatically increases as the body size increases in the normal direction to the plane of contact (see Figs. 1 and 2) so that $|u(A') - u(A)| \gg |u(A)|$ as the distance between the points A and A' increases, and the difference becomes dependent on the imposed longitudinal deformation. In the limit case of semi-infinite bodies with wavy surfaces in contact (see for example, Argatov, 2012; Dundurs et al., 1973; Johnson et al., 1985), instead of the pair force/displacement, namely, the pair pressure/strain plays the key role. In other words, for a normal pressure imposed at infinity, the choice of the “far points” will be insensitive with respect to the level of normal strain, provided they are taken a few wave lengths away in the normal direction from the contact site. Finally note that this special case represents an interest for identifying the contact approach in the problem of rough contact (see, e.g., Barber, 2003; Jones, 2004; Kuzkina and Kachanov, 2015).

With regard to numerical analysis of contact interaction, the concept of “far points” implies their specific choice, because as it was shown above, a not arbitrary extension of the body domains (for a fixed pair of “far points”) can be introduced without careful examination. In the spirit of the theory of local contact by H. Hertz, the approach of the “far points” represents the measure of the local contact deformation, since the global deformations of the contacting bodies are neglected in the Hertzian theory. Therefore, while solving the problem of local contact by numerical methods, the global deformations should be estimated as well.

Thus, the concept of “remote points” is, as a rule, legitimate in the class of considered problems, although for certain combinations of elastic constants and the specific weight it may become questionable (particularly in 1-D and 2-D problems).

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Appendix A. Integral equation for the contact pressure density in 3-D case

We denote by $p(\theta)$, $0 \leq \theta \leq \gamma$, the contact pressure density. The equilibrium equation has the form

$$2\pi R^2 \int_0^\gamma p(\tau) \sin \tau \cos \tau d\tau = P \quad (\text{A.1})$$

where P is the resultant force of the body forces (see Eq.(3.15)). In accordance with the solution of the axially symmetric problem of loading of an elastic sphere by a unit point force balanced by the uniformly distributed body forces (Bondareva, 1969), the radial displacement of the surface points is given by the integral

$$u_r(\theta) = 2\pi R^2 \int_0^\gamma g_r(\theta, \tau) p(\tau) \sin \tau d\tau + \alpha \cos \theta \quad (\text{A.2})$$

where

$$\begin{aligned} -4\pi\mu R g_r(\theta, 0) &= \frac{3(1-2\nu)}{4(1+\nu)} \cos \theta + \frac{1}{\pi} H_r(\theta, 0) \\ H_r(\theta, \tau) &= \frac{\pi(1-2\nu)}{2(1+\nu)} + 4(1-\nu) U(1, \theta, \tau) + \operatorname{Re} \int_0^1 \left(\frac{A_r}{y^m} + \frac{1}{y^2} \right) U(y, \theta, \tau) dy \end{aligned} \quad (\text{A.3})$$

Function $U(y, \theta, \tau)$ is expressed in terms of the complete elliptic integral of the first kind $K(k)$ and has the form:

$$\begin{aligned} U(y, \theta, \tau) &= \frac{K(k)}{k} - \frac{\pi}{2} (1 + y \cos \theta \cos \tau) \\ h^2 &= (1-y)^2 + 4y \sin^2 \frac{\theta + \tau}{2}, \quad k^2 h^2 = 4y \sin \theta \sin \tau \end{aligned}$$

Constants A_r and m depend solely on Poisson's ratio:

$$A_r = 8\nu^2 - 8\nu + 1 + i \frac{16\nu^3 - 16\nu^2 - 4\nu + 5}{\sqrt{3-4\nu^2}}, \quad 2m = 1 - 2\nu + i\sqrt{3-4\nu^2}$$

Substituting the expression (A.2) into the contact condition

$$u_r(\theta) = \frac{R}{\cos \theta} - R, \quad 0 \leq \theta \leq \gamma$$

we obtain the integral equation (Bondareva, 1971):

$$-\frac{R}{2\pi\mu} \int_0^\gamma H_r(\theta, \tau) p(\tau) \sin \tau d\tau = v(\theta), \quad 0 \leq \theta \leq \gamma \quad (\text{A.4})$$

with the kernel (A.3) and the right-hand side

$$v(\theta) = -\alpha \cos \theta - R + \frac{R}{\cos \theta} + \frac{(1-2\nu)\gamma_0 R^2}{4(1+\nu)\mu} \cos \theta \quad (\text{A.5})$$

The asymptotic solution of Eq.(A.4) was obtained by Bondareva (1971) using Aleksandrov's method (see the book of Aleksandrov and Pozharskii, 2001), and the asymptotically exact approximate solution of Eq.(A.4) was given by Argatov (2005).

Introducing a small parameter

$$\varepsilon = \tan \frac{\gamma}{2}$$

we put

$$\tan \frac{\theta}{2} = \varepsilon x, \quad \tan \frac{\alpha}{2} = \varepsilon t \quad (\text{A.6})$$

We also introduce the following notation:

$$q(x) = \frac{4p(2 \arctan \varepsilon x)}{\mu(1 + \varepsilon^2 x^2)^{3/2}}, \quad w(x) = -\frac{2\nu(2 \arctan \varepsilon x)}{R(1 + \varepsilon^2 x^2)^{1/2}} \quad (\text{A.7})$$

$$S_r(x, t) = \frac{t}{\sqrt{(1 + \varepsilon^2 x^2)(1 + \varepsilon^2 t^2)}} \left[\frac{1}{\pi} \operatorname{Re} \int_0^1 \left(\frac{A_r}{y^m} + \frac{1}{y^2} \right) U^0(y) dy \right. \\ \left. + \frac{1-2\nu}{2(1+\nu)} - 2(1-\nu) \left(1 + \frac{(1 - \varepsilon^2 x^2)(1 - \varepsilon^2 t^2)}{(1 + \varepsilon^2 x^2)(1 + \varepsilon^2 t^2)} \right) \right]$$

$$\mathcal{G}_1 = \frac{1-\nu}{2\pi}, \quad U^0(y) = U(y, 2 \arctan \varepsilon x, \arctan \varepsilon t)$$

Making the change of variables (A.6), we rewrite Eq.(A.4) in the form

$$\frac{\mathcal{G}_1}{\varepsilon} \int_0^1 q(t) \frac{4t}{x+t} \mathbf{K} \left(\frac{2\sqrt{xt}}{x+t} \right) dt = - \int_0^1 q(t) S_r(x, t) dt + \frac{1}{\varepsilon^2} w(x) \quad (\text{A.8})$$

and the equilibrium equation (A.1) takes the form

$$\varepsilon^2 \int_0^1 q(t) \frac{(1 - \varepsilon^2 t^2)t}{(1 + \varepsilon^2 t^2)^{3/2}} dt = \frac{P}{2\pi R^2 \mu} \quad (\text{A.9})$$

Thus, the integral operator corresponding to the axisymmetric contact problem for an elastic half-space is identified explicitly (see the left-hand side of Eq.(A.8)).

Now, we rewrite Eq.(A.8) in the form

$$\int_0^1 q(t) \frac{4t}{x+t} \mathbf{K} \left(\frac{2\sqrt{xt}}{x+t} \right) dt = u(x) \quad (\text{A.10})$$

where

$$\frac{\mathcal{G}_1}{\varepsilon} u(x) = \frac{1}{\varepsilon^2} w(x) - \int_0^1 q(t) S_r(x, t) dt \quad (\text{A.11})$$

and function $w(x)$ is defined by Eqs. (A.5) and (A.7)₂.

Making use of the previously obtained general solution (see, e.g., Alexandrov and Pozharskii, 2001), we represent the solution $q(x)$ of Eq.(A.10) in the form:

$$q(x) = \frac{F(1)}{\pi\sqrt{1-x^2}} - \frac{1}{\pi} \int_x^1 \frac{F'(s)}{\sqrt{s^2-x^2}} ds \quad (\text{A.12})$$

where

$$\pi F(x) = u(0) + x \int_0^x \frac{u'(z)}{\sqrt{x^2-z^2}} dz \quad (\text{A.13})$$

with $u(x)$ given by (A.11). From the condition that the contact pressure vanishes on the boundary of the contact area, we have $F(1) = 0$, i.e.,

$$u(0) + \int_0^1 \frac{u'(z)}{\sqrt{1-z^2}} dz = 0 \quad (\text{A.14})$$

As implied by the equality (A.14), formula (A.12) can be rewritten as

$$q(x) = -\frac{1}{\pi} \int_x^1 \frac{F'(s)}{\sqrt{s^2-x^2}} ds \quad (\text{A.15})$$

Substituting expression (A.15) into formula (A.9) and integrating by parts in the internal integral, we obtain

$$\frac{1}{\varepsilon^2} \frac{P}{2\pi R^2 \mu} = \frac{1}{\pi} \int_0^1 F(s) \left[\frac{1-\varepsilon^2 s^2}{1+\varepsilon^2 s^2} - \frac{2\varepsilon s}{(1+\varepsilon^2 s^2)^{3/2}} \ln \left(\sqrt{1+\varepsilon^2 s^2} + \varepsilon s \right) \right] ds \quad (\text{A.16})$$

From Eq.(A.15), we derive the following expression for the maximum of the contact pressures (at the pole of the sphere):

$$q(0) = -\frac{1}{\pi} \int_0^1 \frac{F'(s)}{s} ds \quad (\text{A.17})$$

We now construct the two-term (leading term and first correction) asymptotic solution of Eq. (A.12) as $\varepsilon \rightarrow 0$. The following asymptotic expansion can be established (Bondareva, 1971):

$$S_r(x, t) = t \left[-\frac{1}{2} \operatorname{Re}(A_r + 1) \ln \varepsilon \max\{x, t\} + c_0 \right] + O(\varepsilon), \quad \varepsilon \rightarrow 0 \quad (\text{A.18})$$

$$c_0 = \frac{1-2\nu}{2(1+\nu)} - 4(1-\nu) - \frac{1}{2} \operatorname{Re} \left(A_r \int_0^1 \frac{1-y^{2-m}}{1-y} dy \right)$$

Note that $\operatorname{Re} A_r = 2\beta^2$ where $\beta = 1-2\nu$, while the last integral can be expressed in terms of the harmonic number function $H_\alpha = \int_0^1 \frac{1-y^\alpha}{1-y} dy$.

It also follows, from results above (in particular, formula (A.7)), that

$$w(x) = w(0) - 4\varepsilon^2 x^2 + O(\varepsilon^3), \quad \varepsilon \rightarrow 0 \quad (\text{A.19})$$

where

$$w(0) = \frac{2\alpha}{R} - \frac{3\beta P}{8\pi(1-\nu)R^2\mu}$$

According to the Hertzian theory, contact pressure is given by

$$q^H(x) = q_0 \sqrt{1-x^2} \quad (\text{A.20})$$

We consider the expression (A.20) as a first approximation, which can be refined by constructing the two-term asymptotic approximation for the coefficient q_0 . The substitution of (3.37) into the integral equation (A.17) yields an approximate equation for q_0 . Applying the asymptotic formula (A.18) we obtain

$$\int_0^1 \sqrt{1-t^2} S_r(x, t) dt = \left(-\beta^2 \ln(\varepsilon x) + c_0 \right) \int_0^x \sqrt{1-t^2} dt + \int_x^1 \sqrt{1-t^2} \left(-\beta^2 \ln(\varepsilon x) + c_0 \right) dt$$

Performing the integrations, substituting into formula (A.13) and accounting for the asymptotic expansion (A.19) we find

$$\begin{aligned} \frac{\pi \mathcal{G}_1}{\varepsilon} F(x) = & \frac{1}{\varepsilon^2} w(0) - \frac{q_0}{3} \left(-\beta^2 \ln(2\varepsilon) + c_0 + \frac{4}{3} \beta^2 \right) - 8x^2 \\ & + \frac{q_0 x}{6} \beta^2 \left\{ \ln \frac{1+x}{1-x} + \frac{1}{x} \ln(1-x^2) + x + \frac{1}{2} (1-x^2) \ln \frac{1+x}{1-x} \right\} \end{aligned} \quad (\text{A.21})$$

where $\mathcal{G}_1 = (1-\nu)/2\pi$. Taking the limit as $x \rightarrow 1-0$, we obtain

$$\frac{\pi \mathcal{G}_1}{\varepsilon} F(x) = \frac{1}{\varepsilon^2} w(0) - 8 - \frac{q_0}{3} \left(-\beta^2 \ln(4\varepsilon) + c_0 + \frac{5}{6} \beta^2 \right)$$

and neglecting terms of the order of ε^2 in the integrand of (A.16) we finally arrive at

$$\frac{1}{\varepsilon^2} \frac{P}{2\pi R^2 \mu} = \frac{1}{\pi} \int_0^1 F(s) ds$$

Using (A.21), we obtain

$$\frac{\pi^2 \mathcal{G}}{\varepsilon^3} \frac{P}{\pi R^2} = \frac{1}{\varepsilon^2} w(0) - \frac{8}{3} + \frac{q_0}{3} \left\{ \beta^2 \left(\ln(4\varepsilon) - \frac{19}{12} \right) - c_0 \right\}$$

Using again (A.21), in conjunction with Eq. (A.17) yields

$$q_0 = \frac{\varepsilon}{\pi^2 \mathcal{G}_1} \left(\frac{1}{\varepsilon^2} w(0) + 8 \right) \left\{ 1 - \frac{\varepsilon}{3\pi^2 \theta_1} \left[-\beta^2 \ln(2\varepsilon) + c_0 + \beta^2 \left(\frac{19}{12} + \frac{3\pi^2}{16} - \ln 2 \right) \right] \right\}$$

With the account of the condition $F(1)=0$ ensuring that the contact pressure does not have a singularity at the contact boundary, we have

$$\frac{1}{\varepsilon^2} w(0) - 8 - \frac{q_0}{3} \left(-\beta^2 \ln(4\varepsilon) + c_0 + \frac{5}{6} \beta^2 \right) = 0$$

The last three equations interrelate the three unknown quantities ε , α , and q_0 . After some algebra we obtain formula (3.30) for the vertical displacement of the center of the sphere, as a function of the applied load.

Appendix B. Integral equation for the contact pressure density in 2-D case

In accordance with the solution $\mathbf{G}(\mathbf{x}; \mathbf{x}^\tau)$ of the problem of loading the elastic disk Ω by a unit point force $-(\cos \tau \mathbf{e}_1 + \sin \tau \mathbf{e}_2)$ applied at the point $\mathbf{x}^\tau = (R \cos \tau, R \sin \tau)$ and balanced by the uniformly distributed body forces $S^{-1}(\cos \tau \mathbf{e}_1 + \sin \tau \mathbf{e}_2)$, we can represent the solution $\mathbf{u}(\mathbf{x})$ of the original contact problem in the form

$$\mathbf{u}(\mathbf{x}) = \int_{-\psi}^{\psi} \mathbf{G}(\mathbf{x}; \mathbf{x}^\tau) p(\tau) R d\tau + \alpha \mathbf{e}_2 \quad (\text{B.1})$$

where $p(\theta)$, $\theta \in (-\psi, \psi)$, is the contact pressure density. The contact force is

$$P = \int_{-\psi}^{\psi} p(\tau) \cos \tau R d\tau \quad (\text{B.2})$$

Note that P must satisfy the equilibrium equation for the disk Ω (see Eq.(2.14)).

In accordance with formulas (2.10) and (B.1), the radial displacements of the surface points are represented by the integral

$$u_r(\mathbf{x}^\theta) = \int_{-\psi}^{\psi} g_r(\theta - \tau) p(\tau) R d\tau + \alpha \cos \theta \quad (\text{B.3})$$

where $\mathbf{x}^\theta = (R \cos \theta, R \sin \theta)$, while the kernel function is given by

$$8\pi\mu g_r(\theta) = -2(1-\nu)(1-|\theta|) \sin |\theta| + 8(1-\nu) \ln \left(2 \sin \frac{|\theta|}{2} \right) + 5 - 4\nu \quad (\text{B.4})$$

Substituting the expression (B.3) into the contact condition (see formula (2.4))

$$u_r(\mathbf{x}^\theta) = \frac{R}{\cos \theta} - R, \quad |\theta| \leq \psi$$

we obtain the following integral equation:

$$\int_{-\psi}^{\psi} g_r(\theta - \tau) p(\tau) R d\tau = \frac{R}{\cos \theta} - R - \alpha \cos \theta, \quad |\theta| \leq \psi \quad (\text{B.5})$$

To construct an asymptotic solution to Eq.(B.5) in the case of local contact when $\psi \ll 1$, we replace the kernel (B.4) of Eq.(B.5) by the following asymptotic representation:

$$8\pi\mu g_r(\theta) = 8(1-\nu) \ln |\theta| + 5 - 4\nu + O(|\theta|) \quad (\text{B.6})$$

Using (B.6), Eq.(B.5) takes the form

$$\mathcal{G} \int_{-\psi}^{\psi} (\ln|\theta - \tau| + A_0) p(\tau) R d\tau = \alpha - \frac{R\theta^2}{2} \quad (\text{B.7})$$

Now, in view of (B.2), Eq.(B.7) can be rewritten as

$$\mathcal{G} R \int_{-\psi}^{\psi} \ln|\theta - \tau| p(\tau) d\tau = -\alpha - nA_0 P + \frac{R\theta^2}{2} \quad (\text{B.8})$$

Substituting the Hertzian density

$$p(\theta) = \frac{2P}{\pi R \psi} \sqrt{1 - \frac{\theta^2}{\psi^2}} \quad (\text{B.9})$$

into Eq.(B.8), we arrive at the equation

$$\mathcal{G} R \left(\frac{\theta^2}{\psi^2} - \ln \frac{2}{\psi} - \frac{1}{2} \right) = -\alpha - \mathcal{G} A_0 P + \frac{R\theta^2}{2}, \quad |\theta| \leq \psi \quad (\text{B.10})$$

From Eq.(B.10), it follows that

$$\psi^2 = \frac{2\mathcal{G}P}{R}, \quad \alpha = \mathcal{G} R \left(\ln \frac{2}{\psi} - A_0 + \frac{1}{2} \right) \quad (\text{B.11})$$

We note that, in view of the relation $l = \psi R$, Eqs. (B.9) and (B.11) coincide with Eqs. (2.32), respectively.

Table 1. The constants appearing in formula (3.50) for several typical values of Poisson's ratio ν .

ν	0.0	0.1	0.2	0.3	0.4	0.5
K'	0.5	0.9564	1.305	1.5379	1.6486	1.6316
$C'_0 + K'$	0.691	-0.7708	-1.9431	-2.8169	-3.3967	-3.7129

Table 2. The dimensional quantity μ/γ_0 .

Material	Shear Modulus, μ , (GPa)	Density, ρ_0 , ($\times 1000 \text{ kg/m}^3$)	Parameter $\mu/(\rho_0 g)$ ($\times 10^6 \text{ m}$)
Aluminum [Al]	26	2.71	0.979
Aluminum Alloy	26 – 30	2.64 – 2.8	1.005 – 1.093
Brass	36 – 41	8.4 – 8.75	0.437 – 0.478
Bronze; Regular	36 – 44	7.8 – 8.8	0.471 – 0.51
Copper [Cu]	40 – 47	8.94	0.457 – 0.536
Glass	19 – 24	2.4 – 2.8	0.808 – 0.875
Iron (Cast)	32 – 69	7 – 7.4	0.466 – 0.951
Magnesium Alloy	17	1.77	0.98
Monel (67% Ni, 30% Cu)	66	8.84	0.762
Nickel [Ni]	80	8.89	0.918
Nylon; Polyamide	0.75-1	1.1	0.07 – 0.093
Rubber	$2 \cdot 10^{-4} - 10^{-3}$	0.96 – 1.3	$2.126 \cdot 10^{-5}$ $- 7.849 \cdot 10^{-5}$
Steel	75 – 80	7.85	0.975 – 1.04
Titanium [Ti]	40	4.54	0.899
Titanium Alloy	39 – 44	4.51	0.882 – 0.996
Gold [Au]	28.819	19.32	0.152
Silicon [Si]	79.9	2.33	3.499
Silver [Ag]	30	10.49	0.292
Tin [Sn]	15.44	7.310	0.216